IMM 398
September 1973

Courant Institute of
Mathematical Sciences

# Motion of a Vortex in a Rotating Layer of Fluid Between Rigid Concentric Spheres

A. S. Peters

Sponsored by Advanced Research Projects Agency,
ARPA Order No. 2069

Approved for public release; distribution unlimited.

New York University

ARPA Order No.: 2069

IMM 398
September 1973

New York University

Courant Institute of Mathematical Sciences

# MOTION OF A VORTEX IN A ROTATING LAYER OF FLUID BETWEEN RIGID CONCENTRIC SPHERES

A. S. Peters

This research was supported by the Advanced Research Projects Agency under Grant No. DA-ARO-D-31-124-72-G113.

Views and conclusions contained in this study should not be interpreted as representing the official opinion or policy of the Courant Institute of Mathematical Sciences, or of New York University, or of ARPA.

Approved for public release; distribution unlimited.

## 1. Introduction

This preliminary report is concerned with an approximate description of the two dimensional vortical motion in a thin spherical layer of a gravitating, incompressible, inviscid fluid. We suppose that the fluid is contained between an inner rigid ball and an outer rigid concentric sphere and that the fluid, along with the inner spherical surface, has been rotating with constant angular velocity $\omega$ about a polar axis until a reference time $t = 0$. Thereafter we suppose that the motion is due to the gravitational attraction of the rotating ball and to the sudden creation at $t = 0$ of concentrated vortices with axis normal to the spheres.

The general objective is to discuss the subsequent motion of these isolated vortices, including the case for which $\omega = 0$. However, in this report we confine the discussion primarily to the motion of a single concentrated vortex in the northern hemisphere when certain equatorial boundary conditions are imposed. The differential equations for the motion of such a vortex appear below; and they are followed by an approximate description of its geometric path for a period of time of such duration that it is not necessary to use a computer for the numerical integration of the equations.

There are many physical phenomena which are related to the motion of vortices in a rotating fluid; and the analysis of this kind of motion poses various inherently interesting mathematical problems. Here, the investigation was undertaken with the hope

that the results could be used to approximate roughly the path of a hurricane generated in the northern hemisphere.

## 2. Formulation

Let a and a+h be the radii of two rigid concentric spheres which contain a spherical layer of fluid. Suppose that $h > 0$ is small compared with a; and that the inner spnere rotates about its polar axis with constant angular velocity $\omega$. Let the motion of the fluid be referred to the inner sphere. Let $\rho$ denote the distance of a fluid particle from the center of the sphere while $\phi$ and $\theta$ respectively denote its longitude and colatitude. In terms of these coordinates the velocity of a particle relative to the inner rotating sphere is defined by the components

$$u = (\rho \sin\theta)\frac{d\phi}{dt} = \text{tangential component toward the east;}$$

$$v = -\rho\frac{d\theta}{dt} = \text{tangential component toward the north;}$$

$$w = \frac{d\rho}{dt} = \text{radial component.}$$

If the only body force acting is that due to the gravitational attraction of the solid ball defined by the inner sphere, then the basic hydrodynamical equations which define the motion of the fluid are the continuity equation

$$\frac{1}{\rho}\frac{\partial \rho^2 w}{\partial \rho} + \frac{1}{\sin\theta}\left[\frac{\partial u}{\partial \phi} - \frac{\partial(v \sin\theta)}{\partial \theta}\right] = 0$$

and the momentum equations

$$\frac{du}{dt} - \frac{uv \cot \theta}{\rho} + \frac{uw}{\rho} + 2\omega w \sin \theta - 2\omega v \cos \theta = - \frac{1}{\delta_o \rho \sin \theta} \frac{\partial p}{\partial \phi} ;$$

$$\frac{dv}{dt} + \frac{vw}{\rho} + \frac{u^2 \cot \theta}{\rho} + 2\omega u \cos \theta = \frac{1}{\delta_o \rho} \frac{\partial p}{\partial \theta} ;$$

$$\frac{dw}{dt} - \frac{v^2}{\rho} - \frac{u^2}{\rho} - 2\omega u \sin \theta = -g - \frac{1}{\delta_o} \frac{\partial p}{\partial \rho} .$$

In these equations the differential operator with respect to the time means

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{\rho \sin \theta} \frac{\partial}{\partial \phi} - \frac{v}{\rho} \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial \rho} .$$

The symbol $\delta_o$ denotes the constant density of the fluid; p denotes the pressure; and $g\rho$ denotes the gravitational potential modified by the centrifugal effect of rotation.

Since h is small compared with the large radius a of the inner sphere; and since

$$w(\phi, \theta, a+h, t) = w(\phi, \theta, a, t) = 0$$

let us assume that the radial velocity and the radial variation of u and v can be neglected. Let us also assume that the motion is such that the nonlinear terms in the tangential momentum equations can be neglected; and that the radial momentum equation can be replaced with the hydrostatic law

$$p(\phi, \theta, \rho, t) = p(\phi, \theta, a, t) - g\delta_o(\rho - a) .$$

Under these assumptions which characterize the linear shallow water theory; and with the notation

$$\tilde{u}(\phi,\theta,t) = u(\phi,\theta,a,t) ,$$

$$\tilde{v}(\phi,\theta,t) = v(\phi,\theta,a,t) ,$$

$$\tilde{p}(\phi,\theta,t) = p(\phi,\theta,a,t) ;$$

an approximation to the motion is determined by the equations

$$\frac{\partial \tilde{u}}{\partial \phi} - \frac{\partial(\tilde{v} \sin\theta)}{\partial \theta} = 0 ; \tag{2.1}$$

$$\frac{\partial \tilde{u}}{\partial t} - 2\omega \tilde{v} \cos\theta = - \frac{1}{\delta_o a \sin\theta} \frac{\partial \tilde{p}}{\partial \phi} ; \tag{2.2}$$

$$\frac{\partial \tilde{v}}{\partial t} + 2\omega \tilde{u} \cos\theta = \frac{1}{\delta_o a} \frac{\partial \tilde{p}}{\partial \theta} . \tag{2.3}$$

The elimination of $\tilde{p}$ from (2.2) and (2.3) and the use of (2.1) leads to the equation

$$\frac{\partial}{\partial t} \frac{1}{a \sin\theta} \left[ \frac{\partial(\tilde{u} \sin\theta)}{\partial \theta} + \frac{\partial \tilde{v}}{\partial \phi} \right] + \frac{2\omega \tilde{v} \sin\theta}{a} = 0 . \tag{2.4}$$

It should be remarked that the quantity

$$\tilde{\zeta} = \frac{1}{a \sin\theta} \left[ \frac{\partial(\tilde{u} \sin\theta)}{\partial \theta} + \frac{\partial \tilde{v}}{\partial \phi} \right]$$

is the radial component of vorticity when radial variations are neglected. An integration of (2.4) yields

$$\frac{1}{a \sin\theta} \left[ \frac{\partial \tilde{u} \sin\theta}{\partial \theta} + \frac{\partial \tilde{v}}{\partial \phi} \right] + \frac{2\omega}{a} \int_0^t \tilde{v} \sin\theta \, d\tau = \tilde{\zeta}(\phi,\theta,0) \tag{2.5}$$

where $\tilde{\zeta}(\phi,\theta,0)$ represents the vorticity prevalent in the fluid layer at time $t = 0$.

The equations (2.1) and (2.5) are the basic ones for the motion under consideration. They can be studied by introducing the function $\tilde{\psi}(\phi,\theta,t)$ and setting

$$(2.6) \qquad (a \sin\theta)\frac{d\phi}{dt} = \tilde{u} = \frac{1}{a}\frac{\partial\tilde{\psi}}{\partial\theta} ,$$

$$(2.7) \qquad -a\frac{d\theta}{dt} = \tilde{v} = \frac{1}{a \sin\theta}\frac{\partial\tilde{\psi}}{\partial\phi} ;$$

in this way (2.1) is automatically satisfied. Then if we substitute (2.6) and (2.7) in (2.5) we find $\tilde{\psi}$ must satisfy

$$(2.8) \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[(\sin\theta)\frac{\partial\tilde{\psi}}{\partial\theta}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2\tilde{\psi}}{\partial\phi^2} + 2\omega\int_0^t \frac{\partial\tilde{\psi}}{\partial\phi}\,d\tau = a^2\tilde{\zeta}(\phi,\theta,0) .$$

The problem now is to find a function $\tilde{\psi}$ which satisfies certain prescribed conditions along the boundary of a domain $\tilde{D}$ on the reference sphere; and satisfies (2.8) at each point of $\tilde{D}$. Before we consider this problem, however, let us discuss briefly the simpler case in which the spheres are fixed and the contained spherical layer of fluid is motionless until $t = 0$.

## 3. The Case of no Rotation

If $\omega = 0$ the equation (2.8) becomes

$$(3.1) \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial\tilde{\psi}}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\tilde{\psi}}{\partial\phi^2} = a^2\tilde{\zeta}(\phi,\theta,0) .$$

It is interesting to observe that if we use the transformation, (see Fig. 1),

$$z = re^{i\phi} = (a \tan \frac{\theta}{2})e^{i\phi} = x + iy ,$$

then (3.1) can be reduced to the ordinary two dimensional potential equation. For example, suppose we wish to analyze the motion of a concentrated vortex in the zonal layer defined by

$$D : \begin{cases} \alpha < \theta < \beta \\ 0 \leq \phi \leq 2\pi \end{cases}$$

on the reference sphere subject to the requirement that along the boundary the normal velocity v must be zero along $\theta = \alpha$ and $\theta = \beta$. If we use the above transformation the problem can be reduced to solving

$$(3.2) \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \mu \delta(x - x_1)\delta(y - y_1)$$

for the annular domain

$$D : \begin{cases} q_1 = a \tan \frac{\alpha}{2} < r < a \tan \frac{\beta}{2} = q_2 \\ 0 \leq \phi \leq 2\pi \end{cases}$$

with the boundary conditions

$$\psi(q_1, \phi) = \psi(q_2, \phi) = 0 .$$

In equation (3.2) $\mu$ represents the strength of the vortex; $\delta$ stands for the Dirac delta function; and $x_1$, $y_1$ correspond to the position of the vortex on the sphere.

It should be remarked that the annular domain D is a stereographic projection of the zone $\tilde{D}$ as Fig. 1 indicates.

The above observations suggest the possibility of developing a general theory for the motion of concentrated vortices on a fixed sphere. In the course of doing this the author found that it is basically analogous to the Helmholtz-Kirchoff theory for the planar motion of rectilinear vortices.

## 4. Fundamental Solution. Concentrated Vortex on a Rotating Sphere

Let us return now to the equation (2.8) which can be written in the form

$$(4.1) \qquad \mathcal{O}\psi \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial\tilde{\psi}}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\tilde{\psi}}{\partial\phi^2}$$

$$= a^2\tilde{\zeta}(\phi,\theta,0) - 2\omega\int_0^t \frac{\partial\tilde{\psi}}{\partial\phi}\,d\tau \ .$$

This equation can be analyzed in several different ways. One way is to introduce an appropriate Green's function and then present the partial differential equation as an integral equation. The Neumann expansion of the integral equation leads to

$$\tilde{\psi}(\phi,\theta,t) = \tilde{\psi}_0(\phi,\theta) + \sum_{n=1}^{\infty}\frac{(\omega t)^n}{n!}\tilde{\psi}_n(\phi,\theta) \ .$$

It can be shown that the function in this expansion must satisfy

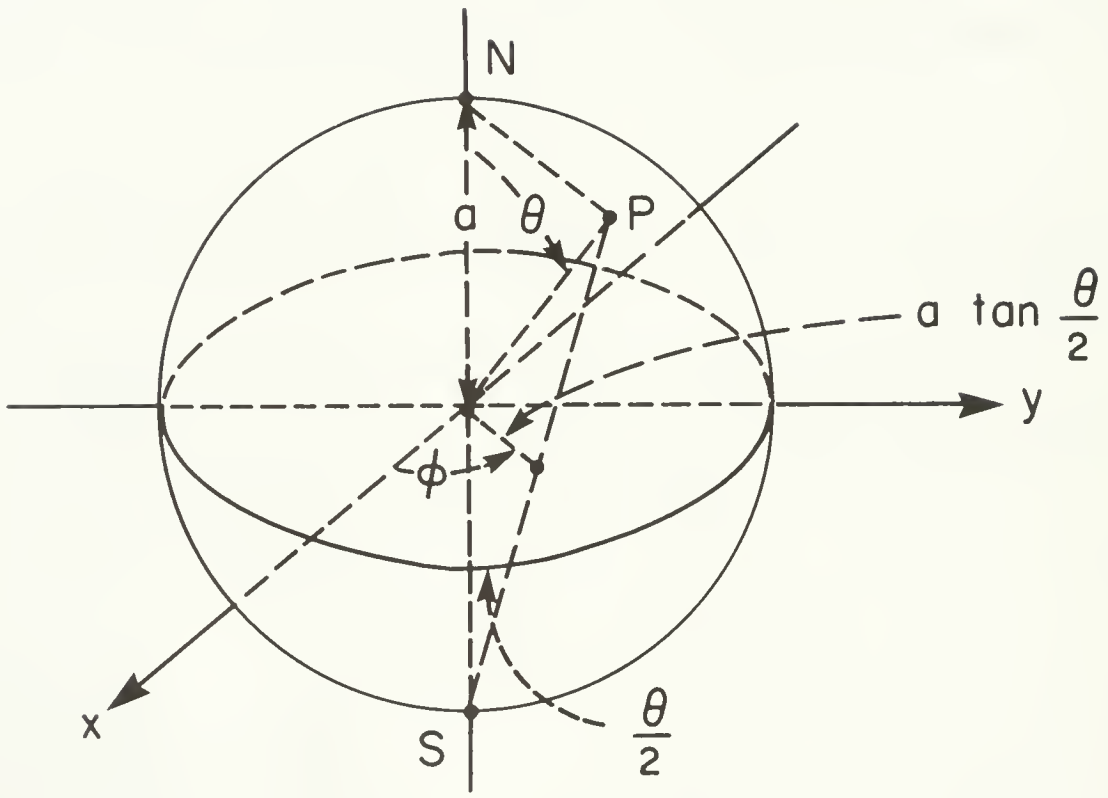


Figure 1

$$\mathcal{O}\tilde{\psi}_0 = a^2\tilde{\zeta}(\phi,\theta,0) ,$$

$$\mathcal{O}\tilde{\psi}_n = -2\frac{\partial}{\partial\phi}\tilde{\psi}_{n-1} ;$$

and therefore, by virtue of the transformation

$$z = (a \tan\frac{\theta}{2})e^{i\phi} ,$$

they can be determined by solving a sequence of ordinary potential problems.

Another way to solve (4.1) is to use transform theory based on finite Fourier transforms in conjunction with Legendre transforms. This way seems most convenient for our immediate aim which is to find the function $\tilde{\psi}$ associated with the creation at $t = 0$ of a vortex of strength $\mu$ concentrated at $(\phi_1,\theta_1)$ where

$$0 < \theta_1 < \pi .$$

In other words, our first object is to solve

$$(4.2)\qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial G}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 G}{\partial\phi^2} + 2\omega\int_0^t \frac{\partial G}{\partial\phi}\,d\tau = \frac{\mu\delta(\phi-\phi_1)\delta(\theta-\theta_1)}{\sin\theta_1}$$

for

$$0 \leq \theta < \pi , \quad 0 \leq \phi \leq 2\pi ;$$

that is, for the whole sphere excluding the south pole where $Q(\phi,\theta,t)$ may be singular.

Let us assume that the behavior of G at the south pole is such that

$$L_{\theta\to\pi} \sin\theta\, G(\phi,\theta,t) = 0 ,$$

$$L_{\theta\to\pi} \sin\theta \frac{\partial G}{\partial\theta} = \text{const.}$$

Under these assumptions it is known that G can be expressed in the form

$$(4.3)\quad G(\phi,\theta,t) = \frac{A(\theta,t)}{} + \frac{1}{2\pi}\sum_{n=1}^{\infty}(2n+1)\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\begin{bmatrix} A_{nm}(t)\cos m\phi \\ +B_{nm}(t)\sin m\phi \end{bmatrix} P_n^m(\cos\theta)$$

where the coefficients are the transforms

$$A(\theta,t) = \int_0^{2\pi} G(\alpha,\theta,t)\,d\alpha ;$$

$$A_{nm}(t) = \int_0^{2\pi}\int_0^{\pi} G(\alpha,\beta,t)\cos m\alpha\, P_n^m(\cos\beta)\sin\beta\, d\beta d\alpha ;$$

$$B_{nm}(t) = \int_0^{2\pi}\int_0^{\pi} G(\alpha,\beta,t)\sin m\alpha\, P_n^m(\cos\beta)\sin\beta\, d\beta d\alpha .$$

The function $P_n^m(\cos\theta)$ is the associated Legendre spherical harmonic of degree n and order m. It can be defined by

$$P_n^m(\cos\theta) = \begin{cases} (-1)^m(\sin\theta)^m\left(\frac{d}{d\cos\theta}\right)^m P_n(\cos\theta) , & m \le n \\ 0 , & m > n \end{cases}$$

and it satisfies

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta)\frac{d}{d\theta}P_n^m(\cos\theta) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]P_n^m(\cos\theta) = 0 .$$

Notice that no previous condition on G is affected if we change G by a constant.

The transform $A(\theta,t)$ must be continuous at $\theta = \theta_1$ and satisfy

$$\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial A}{\partial\theta} = \mu\delta(\theta - \theta_1) .$$

Since $A(\theta,t)$ must be bounded at $\theta = 0$, the solution of the last equation, excepting an additive constant, is uniquely given by

$$A(\theta) = \begin{cases} \mu \ell n \tan\frac{\theta_1}{2}, & \theta \leq \theta_1 , \\ \mu \ell n \tan\frac{\theta}{2}, & \theta_1 \leq \theta . \end{cases} \tag{4.4}$$

This function can be expanded in the form

$$A(\theta) = -\frac{\mu}{2\pi}\ell n \cos\frac{\theta_1}{2} + \frac{\mu}{2\pi}\sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)}[(-1)^n - P_n(\cos\theta_1)]P_n(\cos\theta) \tag{4.5}$$

where $P_n(\ ) = P_n^0(\ )$.

The transforms $A_{nm}(t)$ and $B_{nm}(t)$ must satisfy

$$-n(n+1)A_{nm}(t) + 2\omega m\int_0^t B_{nm}(\tau)d\tau = \mu \cos m\phi_1 \cdot P_n^m(\cos\theta_1) ,$$

$$-n(n+1)B_{nm}(t) - 2\omega m \int_0^t A_{nm}(\tau)d\tau = \mu \sin m\phi_1 \cdot P_n^m(\cos \theta_1)$$

and these yield

$$A_{nm}(t) = -\frac{\mu P_n^m(\cos \theta_1)}{n(n+1)} \cos m\left[\phi_1 - \frac{2\omega t}{n(n+1)}\right],$$

(4.6)

$$B_{nm}(t) = -\frac{\mu P_n^m(\cos \theta_1)}{n(n+1)} \sin m\left[\phi_1 - \frac{2\omega t}{n(n+1)}\right].$$

The substitution of these transforms in (4.3) gives

$$A(\theta)$$

$$(4.7)\quad G(\phi,\theta,t) = -\frac{\mu}{2\pi} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta_1) P_n^m(\cos \theta) \cdot \cos m\left[\phi - \phi_1 + \frac{2\omega t}{n(n+1)}\right].$$

Next, the addition theorem

$$(4.8)\quad P_n[\cos\theta \cos\theta_1 + \sin\theta \sin\theta_1 \cos\lambda]$$

$$= P_n(\cos\theta)P_n(\cos\theta_1) + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta_1)P_n^m(\cos\theta)\cos m\lambda$$

shows that we can write

$$A(\theta)$$

$$(4.9)\quad G(\phi,\theta,t) = - \frac{\mu}{4\pi} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} \left\{ \begin{array}{c} P_n\left[ \begin{array}{c} \cos\theta\cos\theta_1 \\ +\sin\theta\sin\theta_1\cos(\phi-\phi_1+\frac{2\omega t}{n(n+1)}) \end{array} \right] \\ -P_n(\cos\theta_1)P_n(\cos\theta) \end{array} \right\}.$$

Hence, from (4.5), we see that $G(\phi,\theta,t)$ can be exhibited in the form

$$- \frac{\mu}{2\pi} \ell n \cos\frac{\theta_1}{2}$$

$$(4.10)\quad G(\phi,\theta,t) = - \frac{\mu}{4\pi} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} \left\{ \begin{array}{c} P_n\left[ \begin{array}{c} \cos\theta\cos\theta_1 \\ +\sin\theta\sin\theta_1\cos(\phi-\phi_1+\frac{2\omega t}{n(n+1)}) \end{array} \right] \\ -(-1)^n P_n(\cos\theta) \end{array} \right\}.$$

Finally, by using

$$(4.11)\qquad - \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} (-1)^n P_n(x) = 1 + \ell n \frac{(1+x)}{2}$$

we find that

$$- \frac{\mu}{2\pi}\left[\frac{1}{2} + \ell n \cos\frac{\theta_1}{2} + \ell n \cos\frac{\theta}{2}\right]$$

$$(4.12)\quad G(\phi,\theta,t) = - \frac{\mu}{4\pi} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} P_n\left[ \begin{array}{c} \cos\theta\cos\theta_1 \\ +\sin\theta\sin\theta_1\cos(\phi-\phi_1+\frac{2\omega t}{n(n+1)}) \end{array} \right].$$

The reason for retaining the constant $\frac{1}{2} + \ell n \cos \frac{\theta_1}{2}$ is that when we do

$$G(\phi,\theta,t) = G(\phi,\theta,t;\phi_1,\theta_1)$$

is symmetric in $\theta$ and $\theta_1$.

Except for $(\phi_1,\theta_1)$ and the south pole, the velocity components are given by

$$\tilde{u} = \frac{1}{a}\frac{\partial G}{\partial \theta} \; ; \quad \tilde{v} = \frac{1}{a \sin\theta}\frac{\partial G}{\partial \phi} \; .$$

The velocity field in the neighborhood of the concentrated vortex can be estimated by studying the first term of the expansion of $G(\phi,\theta,t)$ in powers of $\omega t$, namely

$$(4.13)\quad G_o(\phi,\theta,t) = -\frac{\mu}{4\pi}\left[1 + 2\,\ell n \cos\frac{\theta_1}{2} + 2\,\ell n \cos\frac{\theta}{2}\right] - \frac{\mu}{4\pi}\sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)} P_n\begin{bmatrix}\cos\theta\cos\theta_1 \\ +\sin\theta\sin\theta_1\cos(\phi-\phi_1)\end{bmatrix}$$

$$= \frac{\mu}{4\pi}\left\{\begin{matrix}-1 - \ell n\cos^2\frac{\theta_1}{2} - \ell n\cos^2\frac{\theta}{2} \\ +1 + \ell n\left[\frac{1 - \cos\theta\cos\theta_1 - \sin\theta\sin\theta_1\cos(\phi-\phi_1)}{2}\right]\end{matrix}\right\}$$

$$= \frac{\mu}{4\pi}\,\ell n\left[\tan^2\frac{\theta}{2} + \tan^2\frac{\theta_1}{2} - 2\tan\frac{\theta}{2}\tan\frac{\theta_1}{2}\cos(\phi-\phi_1)\right] .$$

This gives the dominant part of the velocity of a point near $(\phi_1,\theta_1)$. It corresponds to the case of no rotation; and if we transform to the z-plane introduced above via

$$z = a \tan \frac{\theta}{2} e^{i\phi}$$

we have

$$G_o = \frac{\mu}{4\pi} \ell n\,(z - z_1)(\bar{z} - \bar{z}_1) - \frac{\mu}{4\pi} \ell n\, a^2 \tag{4.14}$$

$$= \frac{\mu}{2\pi} \operatorname{Re}\, \ell n\,(\bar{z} - \bar{z}_1) - \frac{\mu}{2\pi} \ell n\, a$$

where $z_1$ is the image of $(\phi_1, \theta_1)$ and Re $F(z)$ denotes the real part of $F(z)$. The velocity components determined by $G_o$ can be calculated from

$$\tilde{u}_o = \frac{1}{a} \frac{\partial G_o}{\partial \theta} = \frac{\mu(a^2 + z\bar{z})}{4\pi a^2 \sqrt{z\bar{z}}} \operatorname{Re} \frac{z}{z - z_1} \quad ;$$

$$\tilde{v}_o = \frac{1}{a \sin \theta} \frac{\partial G_o}{\partial \phi} = \frac{\mu(a^2 + z\bar{z})}{4\pi a^2 \sqrt{z\bar{z}}} \operatorname{Re} \frac{iz}{z - z_1} \quad .$$

These show that as $z \to z_1$ the velocity approaches infinity; but the component in the direction from $z$ to $z_1$ is zero. Therefore in the neighborhood of $(\phi_1, \theta_1)$ the fluid rotates about the vortex point with velocity of the order of $\frac{1}{|z - z_1|}$. From this, it can be deduced that the velocity field defined by $G$ in the neighborhood of $(\phi_1, \theta_1)$ behaves qualitatively in the same way as the field defined by $G_o$. In other words, the concentrated vortex at $(\phi_1, \theta_1)$ has no tendency to move — it is not self propelling.

Notice that if the vortex is at the north pole, i.e., $\theta_1 = 0$, then

$$G(\phi, \theta, t; \phi_1, 0) = - \frac{\mu}{2\pi} \left[ \frac{1}{2} + \ell n \cos \frac{\theta}{2} \right] - \frac{\mu}{4\pi} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} P_n(\cos \theta)$$

$$G(\phi,\theta,t;\phi_1,0) = G_o(\phi,\theta,t;\phi_1,0)$$

$$= \frac{\mu}{2\pi} \ell n \tan \frac{\theta}{2} .$$

This function is such that

$$L_{\theta\to 0} \sin\theta\, G_\theta(\phi,\theta,t;\phi_1,0) = \frac{\mu}{2\pi} .$$

5. Concentrated Vortex in Northern Hemisphere. Either Velocity Component Zero at the Equator

We proceed to find the stream function for a concentrated vortex traveling in the northern hemisphere and subject to the condition that the normal velocity of the fluid at the equator is zero. The boundary condition then is

$$\tilde{v}(\phi,\tfrac{\pi}{2},t) = \frac{1}{a\sin\theta}\,\tilde{\psi}_\phi(\phi,\tfrac{\pi}{2},t) = 0$$

or

$$\tilde{\psi}(\phi,\tfrac{\pi}{2},t) = c(t) .$$

However, we know that nothing is lost if we impose

$$\tilde{\psi}(\phi,\tfrac{\pi}{2},t) = 0 .$$

The stream function for this case can be formed from $G(\phi,\theta,t;\phi_1,\theta_1)$ if we assume that the principle of reflection across the equator will lead toward the desired result. In fact, it is easy to verify that

(5.1) $$\psi = G(\phi,\theta,t;\phi_1,\theta_1) - G(\phi,\theta,t;\phi_1,\pi-\theta_1)$$

is the desired function. We refrain from a discussion of the velocity field defined by (5.1) in favor of the next case in which the eastward velocity

$$\tilde{u} = \frac{1}{a} \frac{\partial \tilde{\psi}}{\partial \theta}$$

is prescribed at the equator.

We turn to the problem of finding $\psi$ such that with the boundary condition

$$\frac{1}{a} \cdot \psi_\theta(\phi, \tfrac{\pi}{2}, t) = c = \text{const.}$$

the equation

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta) \frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} + 2\omega \int_0^t \frac{\partial\psi}{\partial\phi} \, d\tau = \frac{\mu\delta(\phi - \phi_1)\delta(\theta - \theta_1)}{\sin\theta_1} \tag{5.2}$$

is satisfied for

$$0 \leq \phi \leq 2\pi , \quad 0 \leq \theta < \frac{\pi}{2} .$$

If we assume that

$$L_{\theta\to 0} \sin\theta\psi_\theta = \frac{\mu_o}{2\pi} \tag{5.3}$$

an integration of (5.2) over the northern hemisphere yields

$$\int_0^{2\pi} \psi_\theta(\phi, \tfrac{\pi}{2}, t) d\phi = \mu + \mu_o . \tag{5.4}$$

This shows that the prescription of $\psi_\theta(\phi, \frac{\pi}{2}, t)$ must satisfy the compatability condition (5.4). If we wish to satisfy

$$\psi_\theta(\phi, \tfrac{\pi}{2}, t) = ac$$

then we must have

$$\mu_o = 2\pi ac - \mu \ .$$

As we have seen, the condition (5.3) is satisfied by

$$\frac{\mu_o}{\mu} G(\phi,\theta,t;\phi_1,0) = \frac{\mu_o}{2\pi} \ell n \tan \frac{\theta}{2} \ .$$

Consider the function

$$(5.5) \qquad \psi(\phi,\theta,t) = \frac{\mu_o}{2\pi} \ell n \tan \frac{\theta}{2} + G(\phi,\theta,t;\phi_1,\theta_1) + G(\phi,\theta,t;\phi_1,\pi-\theta_1) \ .$$

A short computation shows that

$$\psi_\theta(\phi, \tfrac{\pi}{2}, t) = \frac{\mu_o}{2\pi} + \frac{\mu}{2\pi} \ .$$

Hence (5.5) satisfies the boundary condition that the eastward velocity at the equator is constant and equal to c if

$$\mu_o = 2\pi ac - \mu \ .$$

We have now shown that

$$(5.6) \qquad \psi(\phi,\theta,t) = \frac{(2\pi ac - \mu)}{2\pi} \ell n \tan \frac{\theta}{2} + G(\phi,\theta,t;\phi_1,\theta_1) + G(\phi,\theta,t;\phi_1,\pi-\theta_1)$$

is the desired stream function.

Let us turn our attention to a study of the path of the vortex at $[\phi_1(t),\theta_1(t)]$. As in the basic Helmholtz theory for

the motion of interacting concentrated vortices, the velocity of the vortex at $[\phi_1(t),\theta_1(t)]$ is equal to the field velocity determined at $[\phi_1,\theta_1]$ by all of the other vortices. That is, the equations for the motion of the vortex are

$$\dot{\phi}_1 = \frac{1}{a^2 \sin\theta_1} \frac{\partial}{\partial\theta} \left[ \begin{matrix} \frac{(2\pi ac - \mu)}{2\pi} \ln \tan\frac{\theta}{2} \\ +G(\phi,\theta,t;\phi_1,\pi-\theta_1) \end{matrix} \right] \Bigg|_{\substack{\phi=\phi_1 \\ \theta=\theta_1}}$$

and

$$\dot{\theta}_1 = -\frac{1}{a^2 \sin\theta_1} \frac{\partial}{\partial\phi} \left[ G(\phi,\theta,t;\phi_1,\pi-\theta_1) \right] \Bigg|_{\substack{\phi=\phi_1 \\ \theta=\theta_1}} .$$

In expanded form these are

$$(5.7)\quad \dot{\phi}_1 = \frac{d\phi_1}{dt} = \frac{(2\pi ac - \mu)}{2\pi a^2 \sin^2\theta_1} + \frac{\mu}{8\pi a^2 \cos^2\frac{\theta_1}{2}}$$

$$- \frac{\mu \cos\theta_1}{4\pi a^2} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} P_n' \left[ \begin{matrix} -\cos^2\theta_1 \\ +\sin^2\theta_1 \cos\frac{2\omega t}{n(n+1)} \end{matrix} \right]$$

$$\cdot \left[ 1 + \cos\frac{2\omega t}{n(n+1)} \right] ;$$

and

$$(5.8)\quad \dot{\theta}_1 = \frac{d\theta_1}{dt} = -\frac{\mu \sin\theta_1}{4\pi a^2} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} P_n' \left[ \begin{matrix} -\cos^2\theta_1 \\ +\sin^2\theta_1 \cos\frac{2\omega t}{n(n+1)} \end{matrix} \right]$$

$$\cdot \sin\frac{2\omega t}{n(n+1)} .$$

It seems that numerical integration is necessary in order to trace the path of the vortex for moderately large or large values of $\omega t$. Our intention is to pursue this in a later report; however, as a preliminary step, we can assume that $\omega t$ is so small that a sufficiently good approximation to the path can be found by expanding $\dot{\phi}_1$ and $\dot{\theta}_1$ in powers of $\omega t$ and retaining those of degree no higher than the second. In addition let us suppose hereafter that $c = 0$.

If each term in (5.7) and (5.8) is expanded in powers of $\omega t$; and if we retain powers no higher than the second we find

$$(5.9)\quad \dot{\phi}_1 = -\frac{\mu}{2\pi a^2 \sin^2\theta_1} + \frac{\mu}{8\pi a^2 \cos^2\frac{\theta_1}{2}}$$

$$- \frac{\mu \cos\theta_1}{2\pi a^2} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} \left\{ \begin{matrix} P_n'(-\cos 2\theta_1) \\ - \frac{(\omega t)^2}{n^2(n+1)^2} \left[ \begin{matrix} P_n'(-\cos 2\theta_1) \\ +2\sin^2\theta_1 \cdot P_n''(-\cos 2\theta_1) \end{matrix} \right] \end{matrix} \right\}$$

and

$$(5.10)\quad \dot{\theta}_1 = -\frac{\mu(\omega t)\sin\theta_1}{2\pi a^2} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} P_n'(-\cos 2\theta_1) .$$

The series which appear in the last equations can be summed and they lead to

$$(5.11)\quad \dot{\phi}_1 = -\frac{\mu}{4\pi a^2(1-\cos\theta_1)\cos\theta_1}$$

$$- \frac{\mu(\omega t)^2}{\pi a^2 \cos\theta_1} \int_0^{\cos\theta_1} \frac{\lambda^3 \ell n\, \lambda}{1-\lambda^2}\, d\lambda ;$$

$$(5.12) \qquad \dot{\theta}_1 = \frac{\mu\omega t \ \ell n \cos \theta_1}{2\pi a^2 \sqrt{1 - \cos^2 \theta_1}} .$$

In order to express $\dot{\phi}_1$ and $\dot{\theta}_1$ in powers of t explicitly we need to expand $\cos \theta_1$ in powers of t and insert the expansion in (5.11) and (5.12). If we do this and retain only powers of t less than the third we find

$$(5.13) \quad \dot{\phi}_1 = - \frac{\mu}{\pi a^2 \cos \theta_o} \left\{ + (\omega t)^2 \left[ \frac{1}{8 \sin^2 \frac{\theta_o}{2}} \int_0^{\cos \theta_o} \frac{\lambda^3 \ \ell n \ \lambda}{1 - \lambda^2} \, d\lambda + \frac{\mu (1 - 2 \cos \theta_o) \ell n \cos \theta_o}{8\pi a^2 \omega \sin^4 \frac{\theta_o}{2} \cos \theta_o} \right] \right\}$$

$$(5.14) \quad \dot{\theta}_1 = \frac{\mu\omega t \ \ell n \cos \theta_o}{2\pi a^2 \sin \theta_o}$$

where $\theta_o$ is the colatitude of the initial position of the vortex.

The approximate equations (5.13) and (5.14) show the following if $\mu$, the strength of the vortex, is positive; and $\theta_o$ is greater than $\pi/3$. When $\omega t$ is very small the vortex moves along a path which is almost parabolic. Since $-a\dot{\theta}_1$ is the velocity to the north, equation (5.14) shows that the rate of change of latitude is zero at t = 0 and the latitude increases as t increases. Equation (5.13) shows that the vortex starts with a westward motion but after a while the positive term

$$- \frac{\mu(\omega t)^2}{\pi a^2 \cos \theta_o} \left[ \int_0^{\cos \theta_o} \frac{\lambda^3 \ell n \lambda}{1 - \lambda^2} \, d\lambda + \frac{\mu(1 - 2 \cos \theta_o) \ell n \cos \theta_o}{8\pi a^2 \omega \sin^4 \frac{\theta_o}{2} \cos \theta_o} \right]$$

becomes dominant and the vortex then moves to the east. The following figure shows a qualitative sketch of the path.

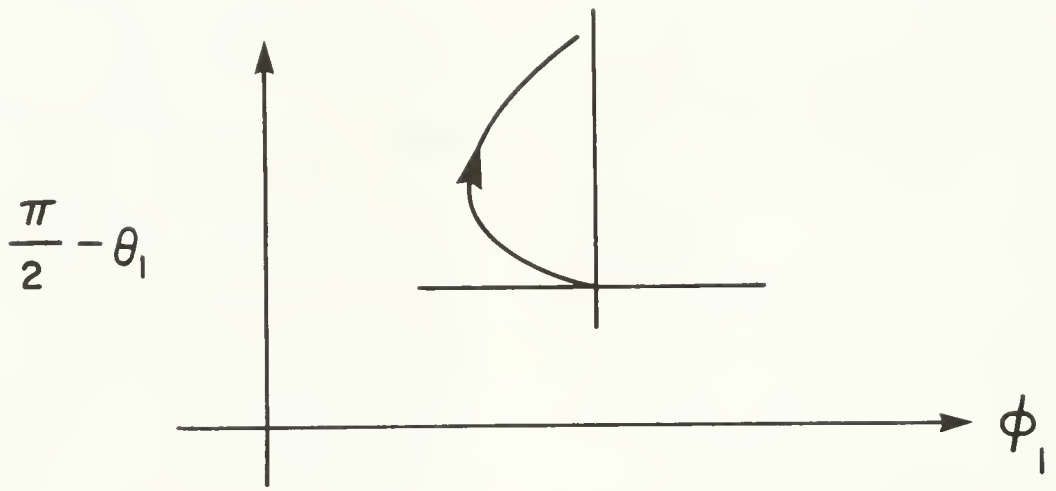


Figure 2

This sketch exhibits some of the characteristics of the observed paths of hurricanes generated in the Caribbean sea or some other southern part of the North Atlantic ocean.

JAN 17 1974

DATE DUE

| | | | |
|---|---|---|---|
| FEB 2 7 1974 | | | |
| JUN 2 2 1979 | | | |
| SEP 1 0 1979 | | | |
| JAN 8 1980 | | | |

GAYLORD PRINTED IN U.S.A.

IMM-398 c.1

Peters

Motion of a vortex in a rotating layer of fluid...

IMM-398 c.1

Peters

AUTHOR

Motion of a vortex in a

TITLE

rotating layer of fluid...

| DATE DUE | BORROWER'S NAME |
|---|---|
| FEB 2 7 1974 | PAV |
| JUN 2 2 1979 | Lavanya |
| | |

N.Y.U. Courant Institute of Mathematical Sciences
251 Mercer St.
New York, N. Y. 10012